H-formulation using the Discontinuous Galerkin method for the 3D Modeling of Superconductors

L. Makong[1,2], A. Kameni[1], F. Bouillault[1], L. Queval[1], P. Masson[2]

[1] GeePs - Group of electrical engineering - Paris, UMR 8507 CNRS, Supelec, Université Paris Sud, Université Pierre et Marie Curie, Plateau du Moulon, 91192 Gif-Sur-Yvette Cedex, France

[2] Department of Mechanical Engineering, University of Houston, Houston, TX 77204, USA

In this paper, a new approach to model efficiently high-temperature superconductors, whose electrical behaviour is characterized by a non-linear power law, in 3-D is investigated. The commonly used non-linear H-formulation will be discretized based on the discontinuous galerkin method. This numerical approach will likely provide fast simulations through parallel computations. Its application on simple examples, such as a superconducting cube subjected to an external magnetic field, will show robustness, convergence and fast computations.

***Index Terms*—High-temperature superconductors, Discontinuous Galerkin Method, H-formulation, Maxwell equations.**

I. Introduction

THE growing interest of high-temperature superconductivity applications, mostly involving machines operating in alternating current [1]-[2], is synonymous with efficient design thus a precise evaluation of AC losses. Numerous difficulties associated with the highly non-linear behaviour of high-temperature superconductors [3], complexities of both the geometry and the magnetic field configuration led to the ongoing development of 3D numerical tools.

However, developed numerical methods [4] mostly using the finite element method are not optimized for the highly scalable computing architecture available nowadays. Thus, the discontinuous galerkin method geared towards parallel computations cangenerate fast and efficient computations of 3D superconductivity problems. Successful implementations of the method have been done to solve the formulation based on the electric field $\mathbf{E}$ [5].

In this paper, the method will be implemented to solve the commonly used $\mathbf{H}$-formulation where $\mathbf{H}$ is the magnetic field. The numerical approach will be thoroughly described and applied to a simulation case involving a superconducting cube subjected to an external sinusoidal magnetic field. Comparisons of computed AC losses with the finite element software in GetDP will be done.

II. Discrete Variational formulation

A typical domain Ω generally consists of a superconducting sub-domain Ω_s and a non-superconducting sub-domain Ω_r that are non-overlapping. Its discretization as $\Omega = \bigcup_{K\in\mathcal{T}_h} K$ will give $\mathcal{T}_h$ the mesh with tetrahedral or hexahedral elements K.

The nodal discontinuous galerkin method will be used to solve the $\mathbf{H}$-formulation resulting from Maxwell equations, the magnetic linear constitutive law and the non-linear electrical power law characterizing the superconducting domain Ω_s. $\mathbf{u}^h$ of the unknown magnetic field $\mathbf{H}$. The approximated quantity $\mathbf{u}^h$ of the unknown magnetic field $\mathbf{H}$ will be defined over each finite element K. It will belong to the following finite element space :

$$\mathbf{W}^h = \left\{\mathbf{w} \in (\mathbf{L}^2(\Omega))^3 : \mathbf{w}|_K \in (\mathbb{P}^m(K))^3, K \in \mathcal{T}_h\right\} \quad (1)$$

Thus the discrete variational formulation will consist in finding $\mathbf{u}^h \in \mathbf{W}^h$ such that:

$$\sum_{K\in\mathcal{T}_h} \int_K \mathbf{u}_t^h \cdot \boldsymbol{\varphi} dK + \sum_{K\in\mathcal{T}_h} \int_K \kappa \cdot \mathrm{curl}\mathbf{u}^h \cdot \mathrm{curl}\boldsymbol{\varphi} dK + I_h = \mathbf{0} \quad , \ \forall \boldsymbol{\varphi} \in \mathbf{W}^h \quad (2)$$

with $\mathbf{u}_t^h = \partial\mathbf{u}^h/\partial t$, $\kappa = \rho/\mu_0$ and the interface term $I_h = -\sum_{K\in\mathcal{T}_h} \int_{\partial K} ((\kappa\cdot \mathrm{curl}\mathbf{u}^h)\times\mathbf{n})\cdot\boldsymbol{\varphi} dA$ where $\mathbf{n}$ is the interface normal vector.

However, the continuity of the tangential components of the magnetic field $\mathbf{H}$ has not been implemented yet. An adapted interface term based on numerical flux expression, which ensures convergence of the problem and includes a constraint term associated with the continuity of the tangential components of $\mathbf{H}$, must be used in place of the interface term I_h.

III. Interface term based on numerical fluxes

The proposed interface term [6], based on the symmetric interior penalty method and numerical fluxes, on each face f belonging to two neighbouring elements K and K' or to the boundary Γ, will be expressed as :

$$-\sum_{f\in\Gamma_h} \int_f [\boldsymbol{\varphi}\times\mathbf{n}] \cdot \{\{\kappa\cdot \mathrm{curl}\mathbf{u}^h\}\} dA - \sum_{f\in\Gamma_h} \int_f [\mathbf{u}^h \times \mathbf{n}] \cdot \{\{\kappa \cdot \mathrm{curl}\boldsymbol{\varphi}\}\} dA + I_h^p \quad (3)$$

with the penalty term $I_h^p = \sum_{f\in\Gamma_h} \int_f \mathtt{a} \cdot [\mathbf{u}^h\times\mathbf{n}]\cdot[\boldsymbol{\varphi}\times\mathbf{n}] dA$.

The quantities $[\mathbf{u}^h \times \mathbf{n}]$ and $\{\{\mathbf{u}^h \times \mathbf{n}\}\}$ denote the jump and average of the tangential components of the field $\mathbf{u}^h$ across each face f.

However, the interface term expression defined above must be simplify in order to implement it numerically. Thus, the expression will be rewritten in terms of fluxes projected on the basis vector function $\boldsymbol{\varphi}$.

The use of the mixed product invariance property will give the interface term expression below where all the projections on $\boldsymbol{\varphi}$ are made :

$$\sum_{f\in\Gamma_h}\int_f [\boldsymbol{\varphi}]\cdot\{\{(\kappa\cdot \mathrm{curl}\mathbf{u}^h)\times\mathbf{n}\}\}dA + \sum_{f\in\Gamma_h}\int_f [(\kappa\cdot \mathrm{curl}\mathbf{u}^h)\times\mathbf{n}]\cdot\{\{\boldsymbol{\varphi}\}\}dA + I_h^p \tag{4}$$

with the penalty term $I_h^p = -\sum_{f\in\Gamma_h}\int_f \mathtt{a}\cdot[\boldsymbol{\varphi}]\cdot[\mathbf{n}\times\mathbf{u}^h\times\mathbf{n}]dA$.

the transformation of the $\mathrm{curl}-\mathrm{curl}$ operator to a divergence operator div will introduce the fluxes quantities. The equivalence of both operators is derived below :

$$\kappa\cdot \mathrm{curl}\mathbf{u}^h = (F_x^h, F_y^h, F_z^h) \tag{5}$$

and

$$\mathbf{F_1} = (0, F_z^h, -F_y^h), \mathbf{F_2} = (-F_z^h, 0, F_x^h), \mathbf{F_3} = (F_y^h, -F_x^h, 0) \tag{6}$$

will give the following derivations

$$\begin{cases} \mathrm{curl}(\kappa\cdot \mathrm{curl}\mathbf{u}^h) = (\mathrm{div}\mathbf{F_1}, \mathrm{div}\mathbf{F_2}, \mathrm{div}\mathbf{F_3}) \\ (\kappa\cdot \mathrm{curl}\mathbf{u}^h)\times\mathbf{n} = (\mathbf{F}_1\cdot\mathbf{n}, \mathbf{F}_2\cdot\mathbf{n}, \mathbf{F}_3\cdot\mathbf{n}) \end{cases} \tag{7}$$

Vectors $\mathbf{F_1}, \mathbf{F_2}$ and $\mathbf{F_3}$ are fluxes quantities. the final interface term, expressed as numerical fluxes, is the following:

$$\sum_{f\in\Gamma_h}\sum_{i=1}^{3}\int_f [\varphi_i]\cdot\{\{\mathbf{F}_i\cdot\mathbf{n}\}\}dA + \sum_{f\in\Gamma_h}\sum_{i=1}^{3}\int_f [\mathbf{F}_i\cdot\mathbf{n}]\cdot\{\{\varphi_i\}\}dA + I_h^p \tag{8}$$

IV. Numerical treatment of the non-linearities arising from the power law

The non-linear resistivity of Ω_s will be derived explicitly or an implicitly.

In the explicit case, the resistivity ρ^{l-1} , evaluated at previous the time step t_p^{l-1} of the problem resolution, will be used as an input in the problem at following time step t_p^l.

In the implicit case, a Newton-Raphson algorithm will ensure internally a good approximation of ρ at each time step t_p^l of the problem resolution.

V. Numerical results

A superconducting cube of $2mm$ side length, subjected to an external magnetic field $\mathbf{H_a} = H_m sin(2\pi f t)\mathbf{e_y}$, will be modeled using the numerical method presented above and the finite element method implemented in GetDP and Comsol. Comparisons of the computed AC losses, over a period, using both methods will help us validate the new approach.

The superconducting behaviour of the cube is characterized by a critical electric field $E_c = 10^{-7}V/mm$, a critical current density $J_c = 100A/mm^2$ and a power law exponent $n = 10$. The magnetic applied flux density amplitude is $B_m = 0.1T$ with a frequency $f = 50Hz$.

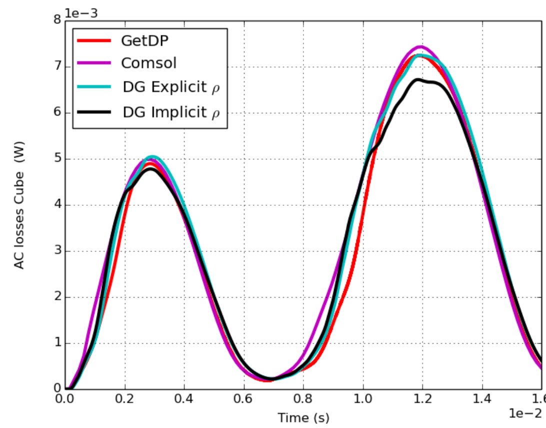

Fig. 1. Total AC losses of a superconducting cube subjected to an external magnetic field computed over T with GetDP, Comsol and the discontinuous galerkin method (DG) with the non-linear resistivity evaluated explicitly and implicitely.

the overall computation time on a mesh of 15948 tetrahedra, with a time step of 25 milliseconds, was 30 minutes for the discontinuous galerkin method using 16 processors and about 2 hours for the finite element method.

Thus, computations using the discontinuous galerkin method are potentially faster than the ones with the finite element method. The reason is parallel computations which are natural to the discontinuous galerkin method. They enable the scaling of the problem resolution over numerous processors.

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